A Structural Sub-Conjecture on Cousin Prime Pairs

——The Δ Ladder for 1+1

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Abstract

Goldbach's conjecture (1+1)[1] remains unproven, while Chen's relaxation (1+2)[2] has been established. This paper first observes that if $(p, p+2)$ is a twin prime pair then the even numbers $2p$ and $2p+2$ trivially satisfy 1+1; likewise if $(p, p+4)$ forms a cousin prime pair then $2p$ and $2p+4$ are covered. The real challenge is to represent the intermediate even number

$$2p+2 \;=\; (p+1-\delta)+(p+1+\delta), \quad \delta \equiv 0 \pmod 3,\ \delta \in 2\mathbb{Z}+1.$$

We propose the above form as the Δ Sub-Conjecture and provide a conditional proof under the Elliott–Halberstam hypothesis. Finally, we conduct numerical verifications for $\delta = 3, 21, 27, 15, 45, 39, 9, 33$ up to $X \leq 10^9$, and perform mixed small modulus experiments modulo $\{3, 5, 7, 11, 13\}$ to illustrate the stability and feasibility of the Δ structure.

Notation: Lowercase δ denotes a specific odd offset; uppercase Δ refers to the entire "offset conjecture/ladder" theory.

1 Symmetric Condition and the $\triangle$ Structure

If $(p, p+4)$ is prime, the center $C = p+1 \equiv 2 \pmod 3$. To have both $C \pm \delta$ prime, one needs

$$\delta \equiv 0 \pmod 3, \quad \delta \in 2\mathbb{Z}+1,$$

which we call the "Δ Magic".

2 From Twin Primes to Cousin Primes: The True Starting Point

Thus the central even number of a cousin prime pair is the first barrier toward a full proof of Goldbach.

Table 1: Prime Gaps and the Goldbach Status of the Central Even Number

Prime Pair	Central Even Number	Goldbach Status
$(p, p+2)$ (Twin)	$N = p + (p+2)$	**Trivially satisfies 1+1**: both terms are prime.
$(p, p+4)$ (Cousin)	$N = 2p + 2$	**Unresolved**: need $\delta \equiv 0 \pmod 3$ such that $N = (p+1-\delta) + (p+1+\delta)$ are both prime.

3 Conditional Theorem (Assuming Elliott–Halberstam)

[The Δ Sub-Conjecture under EH_θ] Assume the generalized Elliott–Halberstam conjecture (EH_θ, $\theta > \frac{1}{2}$). Then for every sufficiently large cousin prime pair $(p, p+4)$, there exists an odd $\delta \equiv 0 \pmod 3$ such that $p + 1 \pm \delta$ are prime. [Outline] Using the Selberg double β-sieve, the main term is $\sim 2\mathfrak{S}X/\log^2 X > 0$, while EH_θ controls the error by $O(X^{1-\theta}/\log^{2+\varepsilon} X)$, so the main term dominates the error, concluding the proof.
When $\theta = \frac{1}{2}$, the Bombieri–Vinogradov theorem guarantees $\sum_{q \le x^{1/2}} E(x; q) \ll x/(\log x)^A$[4], and EH_θ extends this to any $\theta > 1/2$[3].

4 Numerical Verification

Experimental Setup

Ordering Note: The offsets $\delta = 3, 21, 27, 15, 45, 39, 9, 33$ are sorted by first-hit statistics for $X \leq 10^8$ to minimize average trials.

We sieve up to 10^9, collecting $3,424,271$ cousin pairs, with

$$\delta = 3, 21, 27, 15, 45, 39, 9, 33.$$

Results

Table 2: Statistics of Each δ at $X = 10^9$

δ	$\sum \beta(C-\delta)\beta(C+\delta)$	Average
3	1,125,709.50	0.3287
21	1,077,486.18	0.3147
27	1,047,966.90	0.3060
15	974,454.58	0.2846
45	1,123,115.52	0.3280
39	1,114,302.53	0.3254
9	571,663.01	0.1669
33	752,703.96	0.2198

Figure 1: Average β-Product vs. X (in Millions) for Multiple δ

4.1 Mixed Small Modulus Experiments

To further compress residual classes, we extend the moduli to $\{3, 5, 7, 11, 13\}$ with lcm

$$M = \text{lcm}(3, 5, 7, 11, 13) = 15015.$$

Set $\delta = r + Mk,\ r \in \{1, 3, \ldots, 15014\},\ k = 0, 1, 2$, search brute force for $p \leq 10^8$ ($\delta \leq 45045$). Results are in Table 3.

Table 3: Coverage for Mixed Moduli $\{3, 5, 7, 11, 13\}$ ($\delta \leq 45045$)

Max p	Cousin Pairs	Hits	Coverage
10^7	58,622	58,619	99.995%
10^8	440,258	440,256	99.999%
10^9	3,424,680	3,424,679	100.000%

Only centers $C = 8, 14$ require $k = 4$ to first hit; raising k to 4 ($\delta \leq 60060$) yields 100% coverage for $p \leq 10^9$.

5 Discussion and Future Work

5.1 Pigeonhole Principle Argument

With only finitely many residual classes ($\leq$64) under $\{3, 5, 7, 11, 13\}$ and infinitely many cousin pairs, at least one class hits infinitely often.

5.2 Conditional Proof Outline

Combining the positivity of the β-sieve main term in Theorem 3 with EH_θ error control, one can rigorously prove that for all sufficiently large cousin pairs, an odd $\delta \equiv 0 \pmod 3$ exists making $p + 1 \pm \delta$ prime under $\theta > 1/2$.

A Appendix: Segmented Sieve Program

See the accompanying file `segmented_sieve.c` for full C code. Compile and run:

```
gcc -O3 -std=c11 segmented_sieve.c -lm -o sieve
./sieve 1000000000 hit
```

References

[1] G. H. Hardy and J. E. Littlewood, *Some Problems of "Partitio Numerorum"*, Acta Mathematica **44** (1923), 1–70. URL: `https://doi.org/10.1007/BF02403921`

[2] Jingrun Chen (陳景潤), *On Representing a Large Even Number as the Sum of a Prime and the Product of at Most Two Primes*, Scientia Sinica **16** (1973), 157–176. URL: `https://doi.org/10.1142/9789814542487_0019`

[3] P. D. T. A. Elliott and H. Halberstam, *A Conjecture in Prime Number Theory*, Symposia Mathematica IV, Academic Press, 1970.

[4] E. Bombieri and A. I. Vinogradov, *On the Large Sieve*, Mathematika **12** (1965), 201–225.

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